

Metacyclic groups as automorphism groups of compact Riemann surfaces

ANDREAS SCHWEIZER

Abstract

Let X be a compact Riemann surface of genus $g \geq 2$, and let G be a subgroup of $\text{Aut}(X)$. We show that if the Sylow 2-subgroups of G are cyclic, then $|G| \leq 30(g - 1)$. If all Sylow subgroups of G are cyclic, then, with two exceptions, $|G| \leq 10(g - 1)$. More generally, if G is metacyclic, then, with one exception, $|G| \leq 12(g - 1)$. Each of these bounds is attained for infinitely many values of g .

Mathematics Subject Classification (2010): primary 14H37; 30F10; secondary 20F16

Key words: compact Riemann surface; automorphism group; metacyclic group; Z -group; cyclic Sylow subgroup; group of square-free order; exponent

1. Introduction

If G is a finite group, we write $|G|$ for its order and G' for the subgroup generated by all commutators. Moreover, C_n denotes the cyclic group of order n , and D_n is the dihedral group with $|D_n| = 2n$, although some of the sources we use would write D_{2n} for that group.

We are interested in subgroups G of $\text{Aut}(X)$ where X always is a compact Riemann surface of genus $g \geq 2$, and $\text{Aut}(X)$ is its group of conformal (i.e. orientation-preserving) automorphisms. It is a classical result that then $|\text{Aut}(X)| \leq 84(g - 1)$. See for example [Br, Example 3.17]. If $|\text{Aut}(X)| = 84(g - 1)$, and there are infinitely many g for which this can happen, $\text{Aut}(X)$ is called a Hurwitz group.

If one imposes additional conditions on $G \subseteq \text{Aut}(X)$, one usually gets smaller upper bounds on $|G|$. Below, in Theorem 2.3 we recall in detail the known result for the case where G is a metabelian subgroup of $\text{Aut}(X)$. This already shows that if one wants a smooth upper bound such that for infinitely many values of g this bound can be attained, then one should allow finitely many values of g for which there are exceptional groups G that surpass that bound.

In this sense, the optimal bound for solvable subgroups of $\text{Aut}(X)$ is $|G| \leq 48(g - 1)$; see [Ch], [G1]. If $G \subseteq \text{Aut}(X)$ is supersolvable, then $|G| \leq 18(g - 1)$ with one exception for $g = 2$; see [Z2] and corrections in [GM1] and [Z3]. And by [Z1] we have $|G| \leq 16(g - 1)$ if G is nilpotent. The bounds $|G| \leq 4g + 4$ and $|G| \leq 4g + 2$ for abelian respectively cyclic subgroups of $\text{Aut}(X)$ are classical. See also [G1].

This list is not exhaustive. However, there is a class of groups for which only partial results are known, namely the metacyclic groups. See the beginning of Section 4 for some clarifications on the definition of metacyclic.

The best-known metacyclic groups are the dihedral groups, and for these we have $|G| \leq 4g + 4$ by [BCGG, Corollary 2.6]. But certain series of $G \subseteq \text{Aut}(X)$ described in [BJ] (see Theorem 2.4 below) show that for some metacyclic groups $|G|$ can go up to at least $12(g - 1)$.

More generally, for certain types of groups containing a cyclic subgroup of index 2 the paper [Mi] determines the smallest genus on which such a group can act. These formulas are also very useful in our treatment of metacyclic groups with quotient C_4 or C_6 .

For general groups G that have a cyclic quotient that is not too small, [MZ1] determine the smallest genus on which G can act, possibly reversing the orientation. So the smallest genus on which G can act conformally might at worst be bigger.

The non-abelian groups of order pq where p and q are odd primes are obviously split metacyclic. [W] determines every genus on which such a group can act.

In Sections 3 to 5 we determine the optimal upper bounds for $|G|$ in terms of g for four classes of groups: groups with cyclic Sylow 2-subgroups, metacyclic groups, Z -groups (i.e., groups whose Sylow subgroups are all cyclic), and, as a special case of that, groups of square-free order. The main results are Theorems 3.2, 4.9 and 5.5 and Corollary 5.6.

In Section 6 we use the optimal bound for Z -groups to improve the upper bound from [Sch] on the exponent of a solvable group $G \subseteq \text{Aut}(X)$. Actually, the current paper grew out of a remark by the very helpful referee of my paper [Sch], who suggested that it should be possible to find the optimal bound for Z -groups in $\text{Aut}(X)$ by going through the possible signatures of the corresponding Fuchsian groups.

2. Known results

Let X be a compact Riemann surface of genus $g \geq 2$. The universal covering of X is the complex upper halfplane \mathcal{U} (or equivalently the unit disk) and X is the quotient of \mathcal{U} by the group of deck transformations. This generalizes as follows:

If $G \subseteq \text{Aut}(X)$, then there exists a Fuchsian group $\Gamma = \Gamma(h; m_1, m_2, \dots, m_r) \subseteq \text{Aut}(\mathcal{U})$, and a torsion-free, normal subgroup K of Γ , called a surface kernel, such that $\mathcal{U}/K \cong X$, $\Gamma/K \cong G$ and $\mathcal{U}/\Gamma \cong X/G$. One says that $G \subseteq \text{Aut}(X)$ is covered by Γ . Moreover, h is the genus of X/G . The integers m_i are ≥ 2 and called periods.

We refer to [Br, Chapter 1, Section 3] for a survey and only list the facts that we really need. The most important one is

Theorem 2.1. *If G is covered by $\Gamma(h; m_1, m_2, \dots, m_r)$, then*

$$|G| = \frac{2}{2h - 2 + \sum_{i=1}^r (1 - \frac{1}{m_i})} (g - 1).$$

A quick application, that will be very convenient for us, is the following.

Lemma 2.2.

- (a) If $|G| > 4(g - 1)$, then Γ is a Fuchsian group with quotient genus $h = 0$ and $r \in \{3, 4\}$.
- (b) If $|G| > 8(g - 1)$, then $\Gamma = \Gamma(0; m_1, m_2, m_3)$ except possibly for $\Gamma(0; 2, 2, 2, 3)$ which gives $|G| = 12(g - 1)$.

Proof. (a) [Br, Lemma 3.18 (a)].

(b) One easily checks that $\Gamma(0; 2, 2, 2, 4)$ gives $|G| = 8(g - 1)$ and $\Gamma(0; 2, 2, 3, 3)$ gives $|G| = 6(g - 1)$. \square

As an abstract group $\Gamma(h; m_1, m_2, \dots, m_r)$ is generated by $2h + r - 1$ elements with certain relations. Actually, thanks to Lemma 2.2 we only have to do calculations with such groups when $h = 0$. In this case things simplify to

$$\Gamma(0; m_1, m_2, \dots, m_r) \cong \langle x_1, \dots, x_r \mid x_1^{m_1} = \dots = x_r^{m_r} = 1, x_1 x_2 \dots x_r = 1 \rangle.$$

From this one can easily read off Γ/Γ' . This is very important for the corresponding G we are interested in, because G/G' is a quotient of Γ/Γ' . Note however that G' is not necessarily a quotient of Γ' , as G/G' might be a proper quotient of Γ/Γ' .

We now present a known theorem on metabelian automorphism groups. It is representative for all the results that are briefly mentioned in the Introduction. The second reason is that we need some of its details in the proof of Lemma 4.6. Practically all theorems of this form were proved using Fuchsian groups.

When we say that a group G acts on a Riemann surface X , we always mean that it acts effectively, that is, only the neutral element of G acts as identity on X .

Theorem 2.3. [ChP], [G2], [G3] *Let X be a compact Riemann surface of genus $g \geq 2$. Let $G \subseteq \text{Aut}(X)$ be a metabelian group. Then*

$$|G| \leq 16(g - 1)$$

with the following three exceptions:

- a group of order 24 acting on $g = 2$ and covered by $\Gamma(0; 2, 4, 6)$;*
- a group of order 48 acting on $g = 3$ and covered by $\Gamma(0; 3, 3, 4)$;*
- a group of order 80 acting on $g = 5$ and covered by $\Gamma(0; 2, 5, 5)$.*

Conversely, there is a metabelian group of order $16(g - 1)$ acting on a Riemann surface of genus $g \geq 2$ if and only if $g = 2$ or $g = k^2\beta + 1$ is odd where k, β are positive integers such that β divides $1 + \alpha^2$ for some integer α .

Proof. The upper bound and the exceptions for $g = 3$ and $g = 5$ are from [ChP]. The exception for $g = 2$, which had been overlooked in [ChP], is mentioned at the end of [G2]. See also [Br, Example 18.5].

The first proof that the bound $16(g-1)$ is attained for infinitely many g is also in [ChP]. However, half of their g are even; so by the criterion with $g = k^2\beta + 1$ being odd, which is [G3, Theorem 1.1], the corresponding G cannot be metabelian. Thus [G3, Theorem 1.1] corrects and completes the description of the occurring g .

A description of the metabelian groups of order $16(g-1)$ in terms of generators and relations is given in [G3, Theorem 1.2]. \square

If we try to prove an analogue of Theorem 2.3 for groups G that are a semidirect product of two cyclic groups, or for groups G of square-free order, or for groups G that have a quotient C_8 , then the following three known series of groups of automorphisms outline the smallest upper bound for $|G|$ that we can possibly hope for. This will become important in the next three sections.

Theorem 2.4. [BJ, Theorem 1 and Section 3]

- (a) *For each prime $p \equiv 1 \pmod{3}$ there exists a Riemann surface X of genus $p+1$ with $C_{2p} \rtimes C_6 \cong G \subseteq \text{Aut}(X)$. So $|G| = 12(g-1)$. Moreover, G is covered by $\Gamma(0; 2, 6, 6)$.*
- (b) *For each prime $p \equiv 1 \pmod{5}$ there exists a Riemann surface X of genus $p+1$ with $C_p \rtimes C_{10} \cong G \subseteq \text{Aut}(X)$. So $|G| = 10(g-1)$. Moreover, G is covered by $\Gamma(0; 2, 5, 10)$.*
- (c) *For each prime $p \equiv 1 \pmod{8}$ there exists a Riemann surface X of genus $p+1$ with $C_p \rtimes C_8 \cong G \subseteq \text{Aut}(X)$. So $|G| = 8(g-1)$. Moreover, G is covered by $\Gamma(0; 2, 8, 8)$.*

The following three individual groups of automorphism will occur as exceptions in many of the theorems below. Therefore we find it worthwhile to give somewhat detailed descriptions.

Example 2.5. By [N, Theorem 2] there exists exactly one Riemann surface of genus 2 that has an automorphism of order 8, namely $y^2 = x^5 - x$. This is also called the Bolza surface and is known to have automorphism group $GL_2(\mathbb{F}_3)$. Compare [Br, Example 18.5]. The Sylow 2-subgroup is quasidihedral;

$$G \cong \langle a, b \mid a^8 = b^2 = 1, bab^{-1} = a^3 \rangle.$$

Moreover, [Br, Example 18.5] shows that this is the only group of order 16 that can act on genus 2, and that it is covered by $\Gamma(0; 2, 4, 8)$.

Example 2.6. As a special case of [W, Corollary 4.2], the smallest genus on which $G \cong C_7 \rtimes C_3$, the unique non-abelian group of order 21, can act is 3. Now it is known that there are exactly two Riemann surfaces of genus 3 with an automorphism of order 7. The first one is the hyperelliptic $y^2 = x(x^7 - 1)$ (compare [N,

Theorem 1]), whose full automorphism group is C_{14} . The other one is the Klein quartic with projective equation $x^3y + y^3z + z^3x = 0$. It has $\text{Aut}(X) \cong \text{PSL}_2(\mathbb{F}_7)$. Then $G \cong C_7 \rtimes C_3$ is the normalizer of a Sylow 7-subgroup in $\text{Aut}(X)$. That G then is covered by $\Gamma(0; 3, 3, 7)$. Compare the proof of Proposition 4.2 for the last claim.

Example 2.7. By [Br, Example 18.5] the non-abelian group $C_3 \rtimes C_4$ can act on at least one Riemann surface of genus 2. It then is covered by $\Gamma(0; 3, 4, 4)$.

3. Groups with cyclic Sylow 2-subgroups

The following results will be helpful in the next two sections, but they might also be interesting in their own right. We start by recalling a known fact.

Theorem 3.1. *Let G be a finite group with a cyclic Sylow 2-subgroup P . Then the elements of odd order form a normal subgroup N and $G \cong N \rtimes P$. In particular, G is solvable.*

Proof. It is a well-known corollary to Burnside's Transfer Theorem (compare [H, Theorem 14.3.1] or [R, Theorem 10.1.8]) that if the Sylow p -subgroup P for the smallest prime divisor of $|G|$ is cyclic, then it has a normal complement N . This shows $G \cong N \rtimes P$. In our case, N obviously is formed by all elements of odd order.

Now the solvability of G is equivalent to the solvability of N , which is assured by the Theorem of Feit and Thompson [FT]. At the same time this shows that to get the solvability of G one cannot avoid using (or implicitly proving) this monumental theorem. \square

Now we show that for this class of groups there exists an analogue to Theorem 2.3 and the results mentioned in the Introduction.

Theorem 3.2. *Let X be a compact Riemann surface of genus $g \geq 2$. Let G be a subgroup of $\text{Aut}(X)$ such that the Sylow 2-subgroups of G are cyclic. Then*

$$|G| \leq 30(g-1).$$

If G attains this bound then $|G| \equiv 2 \pmod{4}$.

Conversely, there are infinitely many g , necessarily with $g \equiv 6 \pmod{10}$, for which this bound is attained.

Proof. The four biggest possible orders, $84(g-1)$, $48(g-1)$, $40(g-1)$ and $36(g-1)$ are all divisible by 4 but do not allow a quotient of order 4. Compare [Br, Lemma 3.18] or [GM1, Table 4.1]. So $|G| \leq 30(g-1)$.

If this bound is attained, the corresponding Fuchsian group is $\Gamma = \Gamma(0; 2, 3, 10)$, which also only allows an abelian quotient C_2 . Hence $|G| \equiv 2 \pmod{4}$, which is

equivalent to $g - 1$ being odd. Moreover, $\Gamma' = \Gamma(0; 3, 3, 5)$ and $\Gamma'' = \Gamma(0; 5, 5, 5)$. So $G'/G'' \cong C_3$. By [H, Theorem 9.4.2] G''/G''' cannot also be cyclic, so $G''/G''' \cong C_5 \times C_5$. Hence 25 divides $|G|$. So, besides being odd, $g - 1$ is also divisible by 5, i.e., $g \equiv 6 \pmod{10}$.

If in [Ch, Theorem 3.2] we fix $m = 5$ and take for n any odd number, we get a Riemann surface of genus $5 \cdot n^{12} + 1$ with the action of a group of order $|G| = 2 \cdot 3 \cdot 5^2 \cdot n^{12}$. \square

If $|G|$ is divisible by a higher power of 2, we get a smaller bound, even under weaker conditions.

Theorem 3.3. *Let X be a compact Riemann surface of genus $g \geq 2$ and $G \subseteq \text{Aut}(X)$. If G has a quotient C_8 , then $|G| \leq 8(g - 1)$. This bound is sharp by Theorem 2.4 (c).*

Proof. If $|G| \geq 4(g - 1)$ then by Lemma 2.2 the corresponding Fuchsian group must be $\Gamma(0; m_1, m_2, m_3)$ or $\Gamma(0; m_1, m_2, m_3, m_4)$. Moreover, to have a quotient C_8 , at least two of the periods must be divisible by 8. The biggest $|G|$ one can get under these conditions is $|G| = 8(g - 1)$ from $\Gamma(0; 2, 8, 8)$; everything else is smaller. \square

Compare also [MZ1, Theorem 1].

Lemma 3.4. *Let X be a compact Riemann surface of genus $g \geq 2$. If $G \subseteq \text{Aut}(X)$ has a quotient C_4 , then $|G| \leq 10(g - 1)$ except for the following cases:*

- $|G| = 12(g - 1)$ for $\Gamma(0; 3, 4, 4)$;
- $|G| = 16(g - 1)$ for $\Gamma(0; 2, 4, 8)$;
- $|G| = 12(g - 1)$ for $\Gamma(0; 2, 4, 12)$;
- $|G| = \frac{32}{3}(g - 1)$ for $\Gamma(0; 2, 4, 16)$.

Proof. If $|G| > 10(g - 1)$, then by Lemma 2.2 the corresponding Γ must be a triangle group $\Gamma(0; m_1, m_2, m_3)$. Moreover, to have a quotient C_4 , two of the periods must be divisible by 4. This also excludes the exception $\Gamma(0; 2, 2, 2, 3)$.

Now $\Gamma(0; 2, 4, 4)$ is not a triangle group, $\Gamma(0; 3, 4, 4)$ gives $|G| = 12(g - 1)$, $\Gamma(0; 3, 4, 8)$ gives $|G| = \frac{48}{7}(g - 1)$, and $\Gamma(0; 4, 4, 4)$ gives $|G| = 8(g - 1)$. Likewise $\Gamma(0; 2, 4, 8)$ gives $|G| = 16(g - 1)$, $\Gamma(0; 2, 4, 12)$ gives $|G| = 12(g - 1)$, $\Gamma(0; 2, 4, 16)$ gives $|G| = \frac{32}{3}(g - 1)$, and $\Gamma(0; 2, 4, 20)$ gives $|G| = 10(g - 1)$. Finally, $\Gamma(0; 2, 8, 8)$ gives $|G| = 8(g - 1)$. \square

Corollary 3.5. *Let X be a compact Riemann surface of genus $g \geq 2$. If $G \subseteq \text{Aut}(X)$ has cyclic Sylow 2-subgroups and 4 divides $|G|$, then $|G| \leq 12(g - 1)$.*

Proof. If 8 divides $|G|$ this is a Corollary to Theorem 3.3. And if 8 does not divide $|G|$, this excludes the case $|G| = 16(g - 1)$ in Lemma 3.4. \square

Remark 3.6. If the Sylow p -subgroups of G for all *odd* primes p are cyclic, in contrast to Theorem 3.1 this does not suffice to guarantee that G is solvable. Not even if in addition the Sylow 2-subgroup has a cyclic subgroup of index 2. For example, the simple groups $PSL_2(\mathbb{F}_p)$ have this property.

Bounds like in Theorem 3.2 do also not hold for such groups. If $p \equiv \pm 1 \pmod{7}$, then by [Mb, Theorem 8] $PSL_2(\mathbb{F}_p)$ is a Hurwitz group. See [Sch] for more elaborations on this.

4. Metacyclic groups

A finite group G is called **metacyclic** if it has a cyclic normal subgroup N such that G/N is also cyclic. A **split metacyclic** group is a group $G \cong C_m \rtimes C_n$.

The group $C_3 \rtimes C_4$ where the elements of order 4 act as inversion on C_3 has a subgroup $C_3 \times C_2 \cong C_6$ of index 2, but the extension of C_6 by C_4/C_2 is non-split. This shows that a metacyclic group can have several metacyclic structures and that a split metacyclic group can also have non-split metacyclic structures. A metacyclic group that is not split metacyclic is for example the quaternion group Q_8 of order 8.

If $N \triangleleft G$ and G/N are cyclic, this implies of course that $G' \subseteq N$ and hence that G' is cyclic. Some books, e.g. [H, p.146] use the more restrictive definition that a finite group G is metacyclic if G' and G/G' are cyclic. For example, the metacyclic groups $C_p \times C_p$ or D_4 or Q_8 would not be metacyclic in that restrictive sense.

Lemma 4.1. *Every (split) metacyclic group of even order also has a (split) metacyclic structure with a quotient of even order.*

Proof. If $G \triangleright N \cong C_{2^e m}$ and $G/N \cong C_n$ with m, n both odd, then $C_m \triangleleft G$ and $G/C_m = H$ has a normal subgroup C_{2^e} with quotient C_n . By the Schur-Zassenhaus Theorem [R, Theorem 9.1.2] we have $H \cong C_{2^e} \rtimes C_n$. Moreover, $|Aut(C_{2^e})| = 2^{e-1}$; so C_n can only act trivially on C_{2^e} . Thus $H \cong C_{2^e n}$.

By a similar argument we have $C_{2^e m} \rtimes C_n \cong C_m \rtimes (C_{2^e} \rtimes C_n) \cong C_m \rtimes C_{2^e n}$. \square

Contrary to what the proof might seem to suggest upon the first quick reading, we cannot always make the normal cyclic subgroup odd. Just think of $C_2 \times C_2$.

We now embark on proving an analogue of Theorem 2.3 for metacyclic groups.

Proposition 4.2. *Let X be a compact Riemann surface of genus $g \geq 2$ and $G \subseteq Aut(X)$ a metacyclic group of odd order. Then*

$$|G| \leq 9(g-1),$$

except for $G \cong C_7 \rtimes C_3$ acting on genus 3 (see Example 2.6).

Proof. By [MZ2] the only bigger odd orders are $15(g-1)$ and $\frac{21}{2}(g-1)$. In both cases $G/G' \cong C_3$ and G' is a quotient of $\Gamma(0; 5, 5, 5)$ respectively $\Gamma(0; 7, 7, 7)$. As G' must be cyclic, this leaves only the possibilities $|G| = 15$ and $|G| = 21$. But a group of order 15 is cyclic, and hence it cannot act on a Riemann surface of genus 2. \square

The best known (non-abelian) metacyclic groups are of course the dihedral groups, or more generally groups containing a cyclic subgroup of index 2.

Theorem 4.3. *If $G \subseteq \text{Aut}(X)$ contains a cyclic subgroup of odd order and index 2, then*

$$|G| \leq 6g.$$

This bound is attained for any $g \equiv 5 \pmod{6}$ by the groups

$$G \cong \langle a, b \mid a^{3g} = b^2 = 1, b^{-1}ab = a^{g-1} \rangle.$$

Proof. Obviously $G \cong C_n \rtimes C_2$ where n is odd. The minimum genus g^* on which such a group can act was determined in [Mi, Theorem 3.3]. It also depends on the action of C_2 on C_n . If we take G with generators and relations as in our theorem, we get $g^* = \frac{n}{3}$. Some easy estimates show that for the other cases in [Mi, Theorem 3.3] one cannot get a smaller g^* . \square

Theorem 4.4. *If $G \subseteq \text{Aut}(X)$ contains a cyclic subgroup of index 2, then*

$$|G| \leq 8g.$$

This bound is attained for any $g(\geq 2)$ by the groups

$$G \cong \langle a, b \mid a^{4g} = b^2 = 1, b^{-1}ab = a^{2g-1} \rangle.$$

Proof. For the group with generators and relations as in our theorem [Mi, Theorem 3.3] gives g as the minimum genus on which it can act. However, [Mi] only investigates a special type of non-split extensions of C_n by C_2 , and there are still other ones. So to establish the bound on all G that contain C_n of index 2, we instead use that $n \leq 4g + 2$ by [N, Theorem 1]. But, given g , the Riemann surface X of genus g with $C_{4g+2} \subseteq \text{Aut}(X)$ is uniquely determined (still by [N, Theorem 1]), namely it is $y^2 = x(x^{2g+1} - 1)$, and actually $C_{4g+2} = \text{Aut}(X)$. By [N, Theorem 2] the second biggest size of a cyclic group is C_{4g} , which furnishes our upper bound. \square

If G has a bigger cyclic quotient, one also obtains a good bound, even without the assumption that G is metacyclic.

Theorem 4.5. *Let X be a compact Riemann surface of genus $g \geq 2$ and $G \subseteq \text{Aut}(X)$.*

(a) If G has a quotient C_{10} , then $|G| \leq 10(g-1)$. This bound is attained by the groups in Theorem 2.4 (b).

(b) If G has a quotient C_{2p} with a prime $p \geq 7$, then $|G| \leq 7(g-1)$.

Proof. If $|G| > 4(g-1)$, then by Lemma 2.2 G must be covered by a Fuchsian group with quotient genus $h = 0$ and $r \in \{3, 4\}$. Moreover, two periods must be divisible by p and two periods must be even. So the biggest $|G|$ we can get comes from $\Gamma(0; 2, p, 2p)$ and is $\frac{4p}{p-3}(g-1)$. \square

Theorem 4.5 can also be extracted from [MZ1, Theorem 1]. But for quotients C_4 and C_6 only partial results are known. In both cases we establish the bound we want for metacyclic G , but without assuming a priori that the quotient C_4 or C_6 is the one from the metacyclic structure.

Lemma 4.6. *Let $G \subseteq \text{Aut}(X)$ be metacyclic. If G has a (not necessarily cyclic) normal subgroup N with $G/N \cong C_4$, then $|G| \leq 12(g-1)$.*

Proof. By Lemma 3.4 we only have to exclude the possibility $|G| = 16(g-1)$ with Fuchsian group $\Gamma = \Gamma(0; 2, 4, 8)$. This group has $\Gamma/\Gamma' \cong C_2 \times C_4$, so the quotient from the metacyclic structure can only be C_2 or C_4 . But C_2 is excluded by Theorem 4.4, unless $g = 2$. By [Br, Example 18.5] there is only one G of order 16 for $g = 2$. This must be the group in Example 2.5, which obviously has no quotient C_4 . So for the rest of the proof we can assume $g \geq 3$.

Now we know that G must contain a cyclic subgroup of order $4g-4$. By [N, Theorems 3, 4 and 5] this is only possible if $4g-4 \leq 3g+3$, i.e., if $g \leq 7$. But by Theorem 2.3 there are no metabelian groups of order $16(g-1)$ for $g = 7, 6, 4$. And if $g = 5$, then G cannot have an element of order 16 by [N, Theorem 5]. The remaining possibility $g = 3$ is excluded by the following lemma. \square

Lemma 4.7. *A metacyclic group of order 32 cannot act on a Riemann surface of genus 3.*

Proof. By [N, Theorem 1] a Riemann surface of genus 3 cannot have an automorphism of order 16. On the other hand, G cannot have a quotient C_8 by Theorem 3.3. So the metacyclic structure must be $C_8 \cong \langle a \rangle \triangleleft G$ with $G/\langle a \rangle \cong C_4$. Then Lemma 3.4 tells us that G is covered by $\Gamma(0; 2, 4, 8)$. Consequently G/G' is a quotient of $C_2 \times C_4$. Together with $G/\langle a^2 \rangle$ being a group of order 8 with a quotient C_4 this implies $G/\langle a^2 \rangle \cong C_2 \times C_4$ and $G' = \langle a^2 \rangle$.

Furthermore, $G/\langle a^4 \rangle$, having a quotient $C_2 \times C_4$, is neither cyclic nor dihedral, which means that $X/\langle a^4 \rangle$ cannot have genus 0. So a^4 is not a hyperelliptic involution. On the other hand, by the Hurwitz formula $X \rightarrow X/\langle a \rangle$ cannot be a smooth cover, so a^4 must have fixed points on X . Thus a^4 has exactly 4 fixed points and $X/\langle a^4 \rangle$ has genus 1. Moreover, $G/\langle a^2 \rangle \cong C_2 \times C_4$ acts on $X/\langle a^2 \rangle$, which therefore cannot

have genus 0. So $X/\langle a^2 \rangle$ also has genus 1, and hence a^2 has no fixed points.

Obviously, a^4 lies in the center of G . So G acts on the 4 fixed points of a^4 . Let P be one of them. Then the stabilizer $\text{Stab}(P)$ of P in G must have order 8. (Recall that G has no elements of order 16.) So there exists an element $b \in G$ that fixes P with $b^4 = a^4$ but $b^2 \notin \langle a^2 \rangle$ since a^2 and a^6 have no fixed points. From the Hurwitz formula we get that b fixes 2 of the fixed points of a^4 and b^2 fixes all 4.

Obviously a and b generate G , and from $bab^{-1} = a^r$ with $r \in \{1, 3, 5, 7\}$ we get $b^2ab^{-2} = a$. From this it is easy to see that G has exactly 3 elements of order 2, namely a^4 , a^2b^2 and a^6b^2 .

Since G is a 2-group, every point on X with a nontrivial stabilizer in G must be fixed by an involution. As there is at most one hyperelliptic involution, a^2b^2 and a^6b^2 together cannot have more than 12 fixed points. Moreover, if $c \in G$ is an element of order 8, then c^4 lies in $\langle a \rangle$, so $c^4 = a^4$. This means that the fixed points of a^2b^2 or a^6b^2 have stabilizers of order 4 or 2. Summing up $|\text{Stab}(P)| - 1$ for all $P \in X$, we get at most $4 \cdot 7 + 12 \cdot 3 = 64$, not enough for the Hurwitz formula for $X \rightarrow X/G$, which requires 68. \square

Lemma 4.8. *Let $G \subseteq \text{Aut}(X)$ be metacyclic. If G has a (not necessarily cyclic) normal subgroup N with $G/N \cong C_6$, then either $|G| = 12(g-1)$ with G being covered by $\Gamma(0; 2, 6, 6)$ or $|G| \leq 9(g-1)$.*

Proof. If $|G| > 8(g-1)$, then G must be covered by a triangle group $\Gamma(0; m_1, m_2, m_3)$ (Lemma 2.2) and the quotient C_6 implies that at least two periods must be even and at least two periods must be divisible by 3.

If two periods are divisible by 6, we have $\Gamma(0; 2, 6, 6)$ or $|G| \leq 8(g-1)$, as $\Gamma(0; 2, 6, 12)$ gives already $8(g-1)$ and $\Gamma(0; 3, 6, 6)$ gives $6(g-1)$.

Alternatively, the three periods can be divisible by 2, 3 and 6. As $\Gamma(0; 2, 6, 9)$ gives $9(g-1)$ and $\Gamma(0; 3, 4, 6)$ gives $8(g-1)$, there only remains the series

$$\Gamma(0; 2, 3, 6k) \text{ with } k \geq 2, \text{ which gives } |G| = 12 \frac{k}{k-1} (g-1).$$

We have to exclude this series. So let us assume that G is covered by $\Gamma = \Gamma(0; 2, 3, 6k)$. Then $\Gamma/\Gamma' \cong C_6$, and hence G/G' is a quotient of C_6 . In particular, G does not have a quotient of order 4. Since G is metacyclic, this means that 4 cannot divide $|G|$. This also implies that k must be odd.

Taking Theorem 4.3 into account, we must have a metacyclic structure $C_n \cong N \triangleleft G$ and $G/N \cong C_6$. Note that the formula for $|G|$ shows that k divides n . Let $n = p_1^{e_1} \cdots p_s^{e_s}$ with (odd) primes $p_1 < p_2 < \dots < p_s$. If the subgroup C_2 acts as identity on the subgroup $C_{p_i^{e_i}}$ of C_n , one can divide by the other cyclic subgroups of C_n and get a quotient $C_{p_i^{e_i}} \rtimes C_6$ in which C_2 is central. So it has a quotient $C_{p_i^{e_i}} \rtimes C_3$ although the biggest quotient of $\Gamma(0; 2, 3, 6k)$ of odd order can only be C_3 . Thus C_2 must act as inversion on all $C_{p_i^{e_i}}$, and hence on C_n . In other words, G contains a dihedral subgroup D_n of order $2n$ and index 3.

By an analogous reasoning, none of the primes p_i can be congruent to 2 modulo 3. Otherwise $C_3 \subseteq C_6$ would have to act as identity on $C_{p_i^{e_i}}$ and we would get a quotient $D_{p_i^{e_i}}$ of $\Gamma(0; 2, 3, 6k)$.

If p_1^2 divides n or if n is prime, then by [Mi, Corollary 3.4] the smallest genus on which D_n can act is $g^* = n - \frac{n}{p_1} \geq \frac{2n}{3}$. Thus $|G| \leq 9g$. On the other hand, $12(g-1) < 12\frac{k}{k-1}(g-1) = |G|$. This is only possible for $g \leq 3$. So $|G| \leq 27$, and hence $n \leq 4$, i.e., $n = 3$, leading to $k = 3$. Thus $|G| = 18(g-1)$, which is too big for $g = 3$ and does not exist for $g = 2$ [Br, Example 18.5].

If p_1^2 does not divide n and n is also not prime (so $n \geq 3 \cdot 7 = 21$), then by [Mi, Corollary 3.4] the smallest genus on which D_n can act is $g^* = n + 1 - \frac{n}{p_1} - p_1 \geq n + 1 - \frac{n}{3} - 3 = \frac{2n}{3} - 2$, so $|G| \leq 9(g+2)$. On the other hand, $12(g-1) < |G|$. This is only possible for $g \leq 9$. Thus $|G| \leq 99$, giving the contradiction $n \leq 16$. \square

Now we put everything together.

Theorem 4.9. *Let X be a compact Riemann surface of genus $g \geq 2$. If $G \subseteq \text{Aut}(X)$ is metacyclic, then, with the exception of the group of order 16 described in Example 2.5, we have*

$$|G| \leq 12(g-1).$$

This bound is attained by the split metacyclic groups in Theorem 2.4 (a).

Proof. If $|G|$ is odd, see Proposition 4.2.

If $|G|$ is even, then in $C_m \cong N \triangleleft G$ and $G/N \cong C_n$ we can assume that n is even. If n is divisible by an odd prime, we can apply Theorem 4.5 or Lemma 4.8. If 4 divides n , we take Lemma 4.6. Finally, if $n = 2$, then by Theorem 4.4 we have $|G| \leq 8g$, which is $\leq 12(g-1)$ except for $g = 2$. By [Br, Example 18.5] the only possible exception for $g = 2$ is the group of order 16. \square

5. Z-groups

A finite groups whose Sylow subgroups are all cyclic is called a **Z-group**. It is not immediately obvious that such groups are always split metacyclic.

Theorem 5.1. (Zassenhaus) [H, Theorem 9.4.3], [R, Theorem 10.1.10] *A Z-group that is not cyclic can be written as a semidirect product*

$$C_m \rtimes C_n$$

where $(m, n) = 1$ and m is odd. In particular, such a group is split metacyclic.

As Z-groups are metacyclic, the bound in Theorem 4.9 of course holds for them. But the examples we know that reach this bound, namely the groups in Theorem

2.4 (a), are not Z -groups. This becomes clear from the original description in [BJ, Theorem 1], where they are described as G is a split extension of C_p by $C_6 \times C_2$.

On the other hand, the groups in Theorem 2.4 (b) are Z -groups. So we now start to work towards the bound $10(g - 1)$.

Proposition 5.2. Let X be a compact Riemann surface of genus $g \geq 2$, and $G \subseteq \text{Aut}(X)$. If G is a Z -group and 8 divides $|G|$, then $|G| \leq 8(g - 1)$. This bound is attained by the groups from Theorem 2.4 (c).

Proof. This is immediate from Theorem 3.3. \square

Proposition 5.3. If $G \subseteq \text{Aut}(X)$ is a Z -group with $|G| \equiv 4 \pmod{8}$, then, with the exception of $C_3 \rtimes C_4$ acting on genus 2, we have $|G| \leq 10(g - 1)$.

Proof. We still have to eliminate the triangle groups $\Gamma(0; 3, 4, 4)$ and $\Gamma(0; 2, 4, 12)$ from Lemma 3.4. In view of Theorem 4.5 and Lemma 4.8 we have $G \cong C_m \rtimes C_4$ with odd m .

Let us first assume that G is covered by $\Gamma(0; 3, 4, 4)$. If m is divisible by a prime $p \geq 5$, then G has a quotient of order $4p$. But the biggest quotient of $\Gamma(0; 3, 4, 4)$ of order prime to 3 is C_4 . So m must be a power of 3. Since $\text{Aut}(C_{3^e}) \cong C_{2 \cdot 3^{e-1}}$, the subgroup C_2 in G is central. Dividing it out, we get a quotient D_{3^e} . In that group every element order divides either 3^e or 2. So D_{3^e} being a quotient of $\Gamma(0; 3, 4, 4)$ means that it is generated by two elements of order 2 whose product has order 3. But that only gives D_3 . So $e = 1$ and $G \cong C_3 \rtimes C_4$, with $g = 2$ because of $|G| = 12(g - 1)$.

Now assume that G is covered by $\Gamma(0; 2, 4, 12)$. Obviously G has a quotient $H \cong C_3 \rtimes C_4$, which cannot be cyclic by Lemma 4.8. Since H has only one involution, an element of order 4 and an element of order 2 together only generate a group of order 4. So H and hence G cannot be a quotient of $\Gamma(0; 2, 4, 12)$. \square

Proposition 5.4. If $G \subseteq \text{Aut}(X)$ is a Z -group with $|G| \equiv 2 \pmod{4}$, then $|G| \leq 10(g - 1)$.

Proof. If G has a quotient C_{2p} with a prime $p \geq 5$, this follows from Theorem 4.5. If G has a quotient C_6 , the condition $|G| \equiv 2 \pmod{4}$ excludes the possibility $|G| = 12(g - 1)$ in Lemma 4.8. There remains the case that G contains a cyclic group of odd order and index 2. Then Theorem 4.3 yields $|G| \leq 6g$, which can only be bigger than $10(g - 1)$ if $g = 2$. But if $g = 2$, the conditions $|G| \leq 12$ and $|G| \equiv 2 \pmod{4}$ also give $|G| \leq 10$. \square

Putting Propositions 5.2, 5.3, 5.4 and 4.2 together, we get the main result of this section.

Theorem 5.5. *Let X be a compact Riemann surface of genus $g \geq 2$. If $G \subseteq \text{Aut}(X)$ is a Z -group, then, with the exception of $C_3 \rtimes C_4$ acting on genus 2 and $C_7 \rtimes C_3$ acting on genus 3 (see Examples 2.6 and 2.7), we have*

$$|G| \leq 10(g - 1).$$

This bound is attained by the groups in Theorem 2.4 (b).

As groups of square-free order must obviously be Z -groups, we also immediately obtain

Corollary 5.6. *Let X be a compact Riemann surface of genus $g \geq 2$. If $G \subseteq \text{Aut}(X)$ is a group of square-free order, then, with the exception of $C_7 \rtimes C_3$ acting on genus 3, we have*

$$|G| \leq 10(g - 1).$$

This bound is attained by the groups in Theorem 2.4 (b).

6. More on the exponent of G

The exponent $\exp(G)$ of a finite group G is the least common multiple of all element orders.

In [Sch, Theorem 4.4] we showed that for $G \subseteq \text{Aut}(X)$ the optimal upper bound on $\exp(G)$ is $42(g - 1)$. For the case of solvable G we got a smaller bound in [Sch, Proposition 5.3]. For that we needed an upper bound for the order of a Z -group G in $\text{Aut}(X)$ (compare [Sch, Proposition 5.1]).

Recall in this context the easy fact that $\exp(G) = \prod \exp(G_p)$ where the product is over the different primes that divide $|G|$ and G_p is a Sylow p -subgroup of G . In particular, $\exp(G) = |G|$ if and only if G is a Z -group.

Now that we have determined the optimal bound for Z -groups, we can improve the bound on $\exp(G)$ for solvable G . We start with two special cases.

Lemma 6.1. *Let X be a compact Riemann surface of genus $g \geq 2$, and let G be a solvable subgroup of $\text{Aut}(X)$.*

(a) *If $|G| = 30(g - 1)$, then $\exp(G)$ divides $6(g - 1)$.*

(b) *If $|G| = 21(g - 1)$, then $\exp(G)$ divides $3(g - 1)$.*

Proof. If $|G| = 30(g - 1)$, then G is covered by $\Gamma(0; 2, 3, 10)$. So $G/G' \cong C_2$ and G' is covered by $\Gamma(0; 3, 3, 5)$. This implies $G'/G'' \cong C_3$ and G'' is a quotient of $\Gamma(0; 5, 5, 5)$. As G''/G''' cannot also be cyclic by [H, Theorem 9.4.2], we necessarily have $G''/G''' \cong C_5 \times C_5$. So the Sylow 5-group of G cannot be cyclic, and the claim follows.

The proof for (b) is practically the same. \square

Proposition 6.2. *Let $G \subseteq \text{Aut}(X)$. If G is solvable and the genus g of X is bigger than 2, then*

$$\exp(G) \leq 12(g-1).$$

Proof. If $\exp(G) = |G|$, then G is a Z -group, and hence $\exp(G) \leq 10(g-1)$ by Theorem 5.2, except for $G \cong C_7 \rtimes C_3$ on genus 3.

If $\exp(G) = \frac{1}{2}|G|$, then $|G|$ cannot be $48(g-1)$, $40(g-1)$ or $36(g-1)$ because by [Sch, Proposition 5.3] we have $\exp(G) \leq 16(g-1)$. Lemma 6.1 excludes the remaining two solvable possibilities from [GML, Table 4.1] with $|G| > 24(g-1)$.

If $\exp(G) = \frac{1}{3}|G|$, then the Sylow 2-subgroups of G are cyclic, and hence $\exp(G) \leq \frac{1}{3}30(g-1) = 10(g-1)$ by Theorem 3.2.

If $\exp(G) \leq \frac{1}{4}|G|$, then $\exp(G) \leq \frac{1}{4}48(g-1) = 12(g-1)$. \square

Remark 6.3. We don't know whether the bound in Proposition 6.2 is sharp. But more precisely, the proof shows that for solvable G with $g \geq 3$ the only possibilities with $\exp(G) > 10(g-1)$ are $C_7 \rtimes C_3$ on genus 3 and perhaps $\exp(G) = 12(g-1)$ for $|G| = 48(g-1)$ or $|G| = 24(g-1)$.

To see a bit more, let

$$\text{rad}(|G|) = \prod_{p \text{ divides } |G|} p,$$

be the product of the different prime divisors of $|G|$, each one taken only with multiplicity one. Obviously $\text{rad}(|G|)$ divides $\exp(G)$ and $\exp(G)$ divides $|G|$.

Theorem 6.4. *Let X be a compact Riemann surface of genus $g \geq 2$ and let $G \subseteq \text{Aut}(X)$ be solvable. Then, except for $G \cong C_7 \rtimes C_3$ on $g = 3$, we have*

$$\text{rad}(|G|) \leq 10(g-1).$$

This bound is attained by the groups from Theorem 2.4 (b).

Proof. If $\text{rad}(|G|) = |G|$, then $|G|$ is square-free and we can invoke Corollary 5.6. So we only have to show that for solvable G with $|G| > 20(g-1)$ we have $\text{rad}(|G|) \leq 10(g-1)$. By [GML, Table 4.1] for solvable groups the orders bigger than $20(g-1)$ are $48(g-1)$, $40(g-1)$, $36(g-1)$, $30(g-1)$, $24(g-1)$, and $21(g-1)$. For four of them it is obvious that $\text{rad}(|G|) \leq 10(g-1)$; the remaining two were treated in Lemma 6.1. \square

Remark 6.5. If $G \subseteq \text{Aut}(X)$ is not solvable, we have

$$\text{rad}(|G|) \leq 42(g-1)$$

from [Sch, Theorem 4.1]. Also by [Sch, Theorem 4.1], this bound is attained if and only if G is a Hurwitz group of exponent $42(g-1)$ and this number is square-free. For $g > 3$ the first condition is equivalent to $G \cong PSL_2(\mathbb{F}_p)$ with a prime $p \equiv \pm 1 \pmod{7}$ and $g = \frac{p^3-p}{168} + 1$. (See [Sch, Theorem 4.4]). The second condition then is equivalent to p being congruent to ± 3 modulo 8 and $\frac{p-1}{2}$ and $\frac{p+1}{2}$ both being square-free. So the question whether this bound is attained infinitely often is equivalent to the number-theoretic question whether there are infinitely many primes p that are congruent to ± 13 or to ± 27 modulo 56 such that $\frac{p-1}{2}$ and $\frac{p+1}{2}$ are both square-free.

References

- [BJ] M. Belolipetsky and G. A. Jones: Automorphism groups of Riemann surfaces of genus $p + 1$, where p is prime, *Glasgow Math. J.* **47** (2005), 379-393
- [Br] T. Breuer: *Characters and Automorphism Groups of Compact Riemann Surfaces*, LMS Lecture Notes 280, Cambridge University Press, Cambridge, 2000
- [BCGG] E. Bujalance, F.J. Cirre, J.M. Gamboa and G. Gromadzki: On compact Riemann surfaces with dihedral groups of automorphisms *Math. Proc. Cam. Phil. Soc.* **134** (2003), 465-477
- [Ch] B. P. Chetiya: On genres of compact Riemann surfaces admitting solvable automorphism groups, *Indian J. Pure Appl. Math.* **12** (1981), 1312-1318
- [ChP] B. P. Chetiya and K. Patra: On metabelian groups of automorphisms of compact Riemann surfaces, *J. London Math. Soc.* **33** (1986), 467-472
- [FT] W. Feit and J. G. Thompson: Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 775-1029
- [G1] G. Gromadzki: Maximal groups of automorphisms of compact Riemann surfaces in various classes of finite groups, *Rev. Real Acad. Cienc. Exact. Fis. Natur. Madrid* **82 no. 2** (1988), 267-275
- [G2] G. Gromadzki: On soluble groups of automorphism of Riemann surfaces, *Canad. Math. Bull.* **34** (1991), 67-73
- [G3] G. Gromadzki: Metabelian groups acting on compact Riemann surfaces, *Rev. Mat. Univ. Complut. Madrid* **8 no. 2** (1995), 293-305
- [GML] G. Gromadzki and C. Maclachlan: Supersoluble groups of automorphisms of compact Riemann surfaces, *Glasgow Math. J.* **31** (1989), 321-327
- [H] M. Hall: *The Theory of Groups*, Macmillan, New York, 1959

- [Mb] A. M. Macbeath: Generators of the Linear Fractional Groups, in: 1969 Number Theory (*Proc. Sympos. Pure Math.* vol. XII, Houston, Tex., 1967) pp. 14-32
- [MZ1] C. May and J. Zimmerman: The symmetric genus of metacyclic groups, *Topology and its Applications* **66** (1995), 101-115
- [MZ2] C. May and J. Zimmerman: The symmetric genus of groups of odd order, *Houston J. Math.* **34 no. 2** (2008), 319-338
- [Mi] G. Michael: Metacyclic groups of automorphisms of compact Riemann surfaces, *Hiroshima Math. J.* **31** (2001), 117-132
- [N] K. Nakagawa: On the orders of automorphisms of a closed Riemann surface, *Pacific J. Math.* **115 no. 2** (1984), 435-443
- [R] D. J. S. Robinson: *A Course in the Theory of Groups*, Springer GTM 80, New York - Berlin, 1982
- [Sch] A. Schweizer: On the exponent of the automorphism group of a compact Riemann surface, *Arch. Math. (Basel)* **107** (2016), 329-340
- [W] A. Weaver: Genus spectra for split metacyclic groups, *Glasgow Math. J.* **43** (2001), 209-218
- [Z1] R. Zomorrodian: Nilpotent automorphism groups of Riemann surfaces, *Trans. Amer. Math. Soc.* **288 no. 1** (1985), 241-255
- [Z2] R. Zomorrodian: Bounds for the order of supersoluble automorphism groups of Riemann surfaces, *Proc. Amer. Math. Soc.* **108 no. 3** (1990), 587-600
- [Z3] R. Zomorrodian: On a theorem of supersoluble automorphism groups, *Proc. Amer. Math. Soc.* **131 no. 9** (2003), 2711-2713

ANDREAS SCHWEIZER

Department of Mathematics,

Korea Advanced Institute of Science and Technology (KAIST),

Daejeon 305-701,

South Korea

e-mail: `schweizer@kaist.ac.kr`